Designing Reinforced Concrete Rectangular Columns for Biaxial Bending

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Introduction

This report contains design recommendations for estimating the strength of rectangular columns subject to biaxial eccentric loads. Nominal strength values $N_P$ and $N_M$ are readily available in charts, graphs, or in spreadsheet program design aids for loads that act along a principal column axis. Since there are an infinite number of skew directions for biaxial bending, it is not feasible to construct charts, graphs or simple spreadsheet programs for general cases of biaxial eccentric loads on rectangular concrete columns. An overview of design considerations for reinforced concrete columns under biaxial loading was presented in a paper that included advice for development of computer programs specifically to accommodate building code requirements. That overview also described a method for adapting standard uniaxial load design aids for estimating the strength of sections to resist biaxial loading.

This report contains design examples without the use of software specifically developed for biaxial bending of columns. For preliminary design and for designs involving biaxial eccentricity, safe and efficient designs can be obtained by using standard uniaxial load tables, graphs or spreadsheets as described in Reference 4. Since biaxial loading governs flexural design strength requirements for most bridge support columns, those who design bridges should develop or acquire computer software that accurately accommodates biaxial eccentricity loading.

ACI 318 Code Requirements

Section 10.2.3 of ACI 318 specifies that one of the strength limits for concrete sections is when the concrete compressive strain reaches 0.003. Furthermore, Section 10.2.6 and Section 10.2.7 specifies the use of a rectangular stress block for evaluating flexural strength. Column design aid charts and tables, consistent with ACI 318 requirements, are readily available for circular columns and for rectangular columns bent about a principal axis. Round columns with biaxial bending are easily solved with standard charts or tables, since circular sections are assumed to possess the same bending strength for any skew direction. In contrast, determining the strength of biaxially loaded rectangular columns is too complex and extensive to permit production of standard design aid strength data for the unique interaction diagram appropriate for any skewed axis of bending.

However, the ACI Code requirements for biaxially loaded rectangular columns can be satisfied by using the following steps:

1. Assign a strain of 0.003 at one corner of a rectangular section.
2. Locate a neutral axis oriented at an angle from the principal axis. Call $e$ the distance from the neutral axis to the corner at which the strain is 0.003.
3. Use the plane of (failure) strains defined in Step 1 and in Step 2 to determine the strain, the stress, and the force at each longitudinal bar.
4. Define the concrete rectangular compression stress block area bounded by the edges of the cross section and a line parallel to the neutral axis and located at a distance $(1 - \beta)c$ from the neutral axis. Compute, for a concrete stress of 0.85, the resultant concrete force and its distance from each principal axis of the section.
5. The nominal axial strength, $P_n$, is the algebraic sum of longitudinal forces on all the reinforcing bars plus the resultant force on the concrete. The nominal moment strength, $M_{nx}$, about the x-axis is the sum of moments of bar forces about the x-axis plus the concrete resultant force with its moment. The nominal moment strength, $M_{ny}$, about the y-axis is the sum of moments of bar forces about the y-axis plus the moment of the concrete resultant force.
6. Repeat Steps 2 through 5 until $P_n$ is greater than $P_u$ when the angle of the neutral axis produces a value of $M_{nx}/M_{ny}$ equal to $M_{ux}/M_{uy}$, and $\phi M_{ux}$ is greater than $M_{ux}$ while $\phi M_{uy}$ is greater than $M_{uy}$.

The 6-Step procedure, iterated several times as required in Step 6, is technically correct in satisfying ACI 318 Code requirements, but it is far too laborious to be used without the help of computer software. Sections
R10.3.6 and R10.3.7 of ACI 318R state that if the magnitude $P_{ui}$ from the “reciprocal load” equation (1) exceeds $P_n/\phi$, the section may be considered adequate.

$$1/P_{ui} = 1/P_{nx} + 1/P_{ny} - 1/P_{no}$$

for which

$P_{ui} = $ nominal axial load strength at given eccentricity 
along both axes

$P_{nx} = $ nominal axial load strength at given eccentricity 
along the x-axis

$P_{ny} = $ nominal axial load strength at given eccentricity 
along the y-axis

$P_{no} = $ nominal axial load strength at zero eccentricity

### Suggested Design Procedure

Reference 4 reported that Eq. (1) produced ratios $P_{est}/P_n$ that averaged 1.116, indicating that Eq. (1) predicts values of strength $P_{est}$ that are more than 11% higher than the average strengths for the specimens tested. Eq. (1) is simple to apply in order to check the strength of a cross section subjected to a biaxially eccentric compression force. However, it provides almost no help for one seeking a cross section to check. A design procedure that uses the axial force $P_{ui}$ and a pseudo resultant moment $M_{ur}$ defined by Eq. (2) can be used for the selection of a rectangular section. This design procedure is demonstrated in detail in Appendix B of Reference 4. A condensed description of the procedure follows.

$$M_{ur} = 1.11(\bar{M}_{ux})^2 + \left(\frac{hM_{uy}}{b}\right)^2$$

for which

$M_{ur} = $ pseudo resultant moment used to select a section

$M_{ux} = $ required moment about the x-axis (the major axis)

$M_{uy} = $ required moment about the y-axis (the minor axis)

$b = $ width of section (dimension parallel with major axis)

$h = $ thickness of section (dimension perpendicular to major axis)

For the purpose of selecting a column cross section, it is assumed that $M_{ur}$ acts about the major axis of the section being sought, and approximately equal amounts of longitudinal bars are placed in each of the four faces. The cross section selected must be shown to produce, with Eq. (1), a value $\phi P_{ui}$ that is greater than the required $P_{ui}$.

Design graphs from Reference 2 or spreadsheet data from Reference 3 can be used to select a cross section, and to determine values of axial strength for application of Eq. (1). The charts in ACI SP-17 were created without $\phi$ factors and with non-dimensional parameters. The charts can be used for the load factors in either the ACI 318 Building Code or the Canadian Code, and for metric units as well as inch-pound units. The ACI SP-17 charts will be used here because they are a readily available source of reinforced concrete column strength data fulfilling requirements of ACI 318.

### Length Effects

Section 10.11.6 of ACI 318 specifies that for columns subjected to biaxial bending, the moment magnifier is to be determined for bending about each principal axis separately. Thus, the moments $M_{ux}$ and $M_{uy}$ would be taken as $\delta M_{ux}$ and $\delta M_{uy}$ in any cases for which the magnification factors $\delta$ are greater than 1.00. Columns in non-sway frames should have end restraint conditions virtually the same for bending about either principal axis. As the same axial force acts about each axis and flexural stiffness is smaller for bending about the minor axis than for bending about the major axis, it should be apparent that a non-sway rectangular column will have a larger value of $\delta$ for bending about the minor axis than for bending about the major axis. Example calculations for moment magnification can be found in current reinforced concrete textbooks.

### Selecting a Section

The proposed procedure for selecting a column section is straightforward and easiest to apply when there is negligible required moment in addition to the required axial force. Section 10.3.6.1 of ACI 318 establishes an upper limit for axial force by applying a reduction coefficient, here designated with the symbol $\alpha_{comp}$, to the theoretical load $P_{no}$. The coefficient $\alpha_{comp} = 0.80$ for tied columns, and $\alpha_{comp} = 0.85$ for spiral columns.

$$\phi P_{n,max} = \alpha_{comp} \left[0.85 f'_c A_s + \frac{A_{st} f_y}{\phi} \right]$$

To get a first estimate for a column cross section of a tied rectangular column with width $= b$ and thickness $= h$, substitute $\phi bh$ for $A_s$ and let $\phi = 0.65$,

$$\phi P_{n,max} = (0.80)0.65 \left[0.85 f'_c + \rho (f_y - 0.85 f'_c) \right] bh$$

$$\phi P_{n,max} = 0.52 \left[0.85 f'_c + \rho (f_y - 0.85 f'_c) \right] bh$$

A desired value of $\rho$, usually between 0.01 and 0.06, can be used in Eq. (4) in order to solve for a required area $bh$. Eq. (4) applies only when $M_s$ is negligible and the minimum eccentricity on the section controls. A “typical” usable strength-interaction diagram as shown in Fig. 1 indicates that the maximum axial force governs only for
conditions of small required moment and small eccentricity 
\( e = \frac{M_u}{P_u} \). If the eccentricity ratio \( \frac{M_u}{P_u} \) is greater than approximately 10% of the section thickness, a coefficient less than 0.52 can be assumed and substituted into Eq. (4) in order to estimate a trial section area \( bh \). The value of \( D \) may be taken as 0.04 near the base of a structure, but smaller \( D \) values generally are more efficient. The minimum allowed value of \( D \) is 0.01. Some iteration with a trial coefficient, \( \phi_\text{comp} \), less than 0.52 can lead to an acceptable section size.

**Example 1 - Using the charts from ACI SP-17**

Given: Grade 60 reinforcement \( f_y = 60 \text{ ksi} \) and \( f'_c = 5.0 \text{ ksi} \). Select reinforcement for a square cross section column to resist an axial force and moments \( P_{ui} = 628 \text{ k}, M_{ux} = 260 \text{ k-ft} \) and \( M_{uy} = 175 \text{ k-ft} \).

1. Compute \( M_u \) using Eq. (2). Recognize that for a square section \( h/b = 1 \).
   \[
   M_u = 1.1 \sqrt{(M_m)^2 + \left(\frac{hM_m}{b}\right)^2} = 1.1 \sqrt{(260)^2 + (175)^2} = 345 \text{ k-ft} = 4,140 \text{ k-in}
   \]

2. Estimate size needed for a square section with \( \rho = 3\% \) longitudinal reinforcement.

   Effective eccentricity of force \( = \frac{M_u}{P_{ui}} = 4,140/628 = 7 \text{ inches} \), an eccentricity that is probably 3 to 4 times larger than an eccentricity near 1.5 in. to 2.5 in., which is the eccentricity at the maximum axial force on a square section between 15 in. to 25 in. thick. Since actual eccentricity \( e \) may be larger than 0.3\( h \), estimate a coefficient \( \phi_\text{comp} \) about half the value \( \phi_\text{comp} = 0.65(0.80) = 0.52 \).

   Thus,
   \[
   P_{ui} = (1/2)\phi_\text{comp}\left[0.85f'_c + \rho(f_y - 0.85f'_c)\right]bh
   \]

3. Solve for \( bh \): \( 628 = 0.5(0.52)\left[0.85(5.0)
   + 0.03(60 - 0.85(5.0))\right]bh \)

   Solving for \( bh \): \( bh = 408 \text{ in}^2 \)

4. Try a section 20 in. by 20 in. for which the ratio \( \gamma = 20 - (2.5) / 20 = 0.75 \).

5. Compute chart coordinates:
   \[
   P_n/\phi = 628/\left[0.65(5.0)400\right] = 0.483 = K_n
   \]
   \[
   M_{ux}/\left[\phi f'_c h\right] = 12(260)/\left[0.65(5.0)400(20)\right] = 0.120
   \]

6. Compute chart coordinates:
   \[
   P_n/\phi = 628/\left[0.65(5.0)400\right] = 0.483 = K_n
   \]
   \[
   M_{ux}/\left[\phi f'_c h\right] = 12(260)/\left[0.65(5.0)400(20)\right] = 0.120
   \]

7. Construct eccentricity lines as demonstrated from origin to coordinates for \((P_{ui}, M_{ux})\) and \((P_{uy}, M_{ux})\). Use ACI SP-17, Chart 7.3.2 to obtain,

   When \( \rho = 0.0254 \), with \( \gamma = 0.7 \), \( P_{ux} = 0.59 f'_c A_g = 0.59(5.0)400 = 1,180 \text{ k} \), and \( P_{uy} = 0.74 f'_c A_g = 0.74(5.0)400 = 1,480 \text{ k} \)

   Also when \( \rho = 0.0254 \),
   \[
   P_{no} = 1.13(5.0)400 = 2,260 \text{ k} \text{ (for all } \gamma \text{ values})
   \]

8. Use ACI SP-17, Chart 7.3.3 to obtain, when \( \rho = 0.0254 \), with \( \gamma = 0.8 \), \( P_{ux} = 0.61 f'_c A_g = 0.61(5.0)400 = 1,220 \text{ k} \), and \( P_{uy} = 0.76 f'_c A_g = 0.76(5.0)400 = 1,520 \text{ k} \)

9. Since \( \gamma = 0.75 \) for thickness \( h = 20 \text{ in.} \), use average of values \( P_{ux} \) and \( P_{uy} \) for Eq. (1):
   \[
   P_{ux} = \frac{(1,180 + 1,220)}{2} = 1,200 \text{ k}
   \]
   \[
   P_{uy} = \frac{(1,480 + 1,520)}{2} = 1,500 \text{ k}
   \]

10. Use the charts from ACI SP-17, Chart 7.3.2 with \( \rho = 0.70 \) and Chart 7.3.3 with \( \rho = 0.80 \).

11. The chart with \( \rho = 0.70 \) indicates a value \( \rho = 0.0254 \) for the computed coordinates. Try 8-#10 bars for which \( D = 8(1.27)/400 = 0.0254 \).
Then \( \phi P_{ui} = 0.65(946) = 615 \text{k.} \) Since 615 k is only 2.1% less than the required \( P_{ui} = 628 \text{k}, \) the section is acceptable.

**Use of Spreadsheet Program for Strength Diagrams**

A spreadsheet program has been created for column strength interaction diagrams.\(^3\) The program uses a generalized section for which the width, thickness, material strengths \( f'_c \) and \( f_y \) can be specified. Also, longitudinal reinforcement areas are specified as the bar area and number of bars in each face. Thus, a section with 8 bars could have 3 bars in each face, or 4 bars in the end faces with 2 bars in the side faces, or 2 bars in the end faces and 4 bars in the side faces. Thickness \( h \) is the distance between end faces. The distance from edge of section to the center of longitudinal bars must be specified by the user. In addition to the maximum axial strength for a concentric compression force, the program computes \( \phi P_n \) and \( \phi M_n \) for six different values of strain in the tension face reinforcement. The strain values specified are:

1) zero moment,
2) half the yield strain of reinforcement,
3) the yield strain of reinforcement,
4) half the sum of yield strain plus 0.005,
5) 0.005 and
6) a strain well in excess of 0.005

The strength-reduction factor \( \phi \) is 0.65 for coordinate points 1), 2), and 3). The value of \( \phi \) for point 4) is 0.775, and \( \phi = 0.90 \) for coordinate points 5) and 6). Interaction diagrams are constructed as straight lines between coordinate points.

An equation for the straight line between any two interaction curve points \((P_q, M_q)\) and \((P_r, M_r)\) can be written:

\[
(P_q - P)/(M_q - M) = (P_r - P)/(M_r - M)
\]  
(5)

See Fig 3. The slope \( S \) of Eq. (5) with reference to the vertical axis is:

\[
S = (M_q - M_r)/(P_q - P_r)
\]

The relationship between \( M \) and \( P \) along any eccentricity line is:

\[
M = Pe
\]

A formula for the value of axial force \( P \) at the intersection of the two lines can be derived by substituting \( Pe \) for \( M \) in Eq. (5) to obtain:

\[
P = (SP_q - M_q)/(S - e)
\]  
(6)
Eq. (6) will be used to compute precise values for axial strengths $\phi P_{\mu x}$ and $\phi P_{\mu y}$ at the required eccentricity $e = M_{u x}/P_{u x}$ or $M_{u y}/P_{u y}$. The precise values of axial strength are to be used in Eq. (1) when $\phi P_{\mu}$ is to be determined.

Example 2 – Use spreadsheet for strength interaction diagrams

Given: Grade 60 reinforcement ($f_y = 60$ ksi) and $f_y' = 5.0$ ksi. Select reinforcement for a square section column to resist axial force $P_{u x} = 628$ k, $M_{u x} = 260$ k-ft and $M_{u y} = 175$ k-ft.

Use the same data as that given with Example 1 in order to determine a trial size. This time, try an 18 in. by 18 in. square section with more reinforcement; say 12-#11 bars with 4 bars in each face.

$$\rho = 12(1.56)/(18 \times 18) = 0.0058$$

Data from the spreadsheet is found in Table 1:

See Fig. 4. Compute eccentricities:

$$e_y = M_{u y}/P_{u y} = 260/628 = 0.414 \text{ ft}$$

$$e_x = M_{u x}/P_{u x} = 175/628 = 0.279 \text{ ft}$$

Precise value $\phi P_{\mu y}$, at $e_y = 0.414$ ft lies between $P_1 = 973$ k, $M_1 = 266$ ft-k and $P_2 = 623$ k, $M_2 = 356$ ft-k:

Compute

$$S = (M_1 - M_2)/(P_1 - P_2)$$

$$= (266 - 356)/(973 - 623) = -0.257 \text{ ft}$$

Compute

$$\phi P_{\mu y} = (SP_1 - M_1)/(S - e_y)$$

$$= [(-0.257)973 - 266]/(-0.257 - 0.279)$$

$$= 963 \text{ k}$$

Precise value $\phi P_{\mu x}$, at $e_x = 0.279$ ft lies between $P_1 = 973$ k, $M_1 = 266$ ft-k and $P_2 = 623$ k, $M_2 = 356$ ft-k:

Compute

$$S = (M_1 - M_2)/(P_1 - P_2)$$

$$= (266 - 356)/(973 - 623) = -0.257 \text{ ft}$$
Table 2 Spreadsheet Data for Example 2 (Grade 75)

<table>
<thead>
<tr>
<th>Point</th>
<th>φM (k-ft)</th>
<th>φP (k)</th>
<th>e (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M₁, P₀</td>
<td>0.00</td>
<td>1755.90</td>
<td>0.000</td>
</tr>
<tr>
<td>0, Pₘₐₓ</td>
<td>0.00</td>
<td>1404.70</td>
<td>0.000</td>
</tr>
<tr>
<td>Mₘₐₓ, Pₘₐₓ</td>
<td>141.87</td>
<td>1404.70</td>
<td>0.101</td>
</tr>
<tr>
<td>P₁, M₁</td>
<td>294.84</td>
<td>1026.00</td>
<td>0.287</td>
</tr>
<tr>
<td>P₂, M₂</td>
<td>387.66</td>
<td>575.38</td>
<td>0.674</td>
</tr>
<tr>
<td>P₃, M₃</td>
<td>445.64</td>
<td>210.69</td>
<td>2.115</td>
</tr>
<tr>
<td>P₄, M₄</td>
<td>499.41</td>
<td>76.04</td>
<td>6.568</td>
</tr>
<tr>
<td>P₅, M₅</td>
<td>544.13</td>
<td>–114.40</td>
<td>–4.756</td>
</tr>
</tbody>
</table>

Rectangular Section 18 x 18 inches
fₑ = 5.0 ksi  ρ = 0.0578  fₘₐₓ = 75.0 ksi

Fig. 6 Interaction Diagram for Example 2 (Grade 75)

Since 618 k is only 1.6% less than the required 628 k, the section is acceptable.

Example 3 – Select reinforcement for a rectangular section.

A rectangular section 18 inches wide and 24 inches thick must resist an axial force P₀ = 543 k and moments
Mₓ = 373 k-ft and Mᵧ = 214 k-ft. Select Grade 60 longitudinal reinforcement for these forces if fₑ = 5,000 psi.

Compute Mₓ using Eq. (2)

\[ Mₓ = 1.1 \sqrt{ \left( Mₓ \right)^2 + \left( hMᵧ/b \right)^2 } \]

\[ = 1.1 \sqrt{ \left( 373 \right)^2 + \left( 24(214)/18 \right)^2 } = 517 \text{ k-ft} \]
Use the spreadsheet column strength program to try various amounts of longitudinal bars until an arrangement is found to be adequate for $P_u = 543$ k when $M_{ur} = 517$ k-ft about major axis.

Use 12-#10 bars, 4 bars in each face. The spreadsheet program gives data as shown in Tables 3 and 4:

Compute eccentricities:
\[ e_x = \frac{M_{ur}}{P_u} = \frac{373/543}{4.00} = 0.687 \text{ ft} \]
\[ e_y = \frac{M_{ur}}{P_u} = \frac{214/543}{4.00} = 0.394 \text{ ft} \]

Precise value $\phi P_{uy}$ at $e_y = 0.687$ ft lies between $P_2 = 779$ k, $M_2 = 521$ ft-k and $P_b = 494$ k, $M_b = 594$ ft-k

Compute
\[ S = \left( \frac{M_2 - M_b}{P_2 - P_b} \right) \]
\[ = \left( \frac{521 - 494}{779 - 494} \right) = -0.256 \text{ ft} \]

Compute
\[ \phi P_{wy} = \left( \frac{SP_2 - M_2}{S - e_y} \right) \]
\[ = \left( \frac{-0.256(779) - 521}{-0.256 - 0.687} \right) = 764 \text{ k} \]

Precise value $\phi P_{wy}$ at $e_y = 0.394$ ft lies between $P_1 = 1,113$ k, $M_1 = 292$ ft-k and $P_2 = 745$ k, $M_2 = 377$ ft-k

Compute
\[ S = \left( \frac{M_1 - M_b}{P_1 - P_2} \right) \]
\[ = \left( \frac{292 - 377}{1,113 - 745} \right) = -0.231 \text{ ft} \]

Compute
\[ \phi P_{wx} = \left( \frac{SP_1 - M_1}{S - e_x} \right) \]
\[ = \left( \frac{-0.231(1,113) - 292}{-0.231 - 0.394} \right) = 877 \text{ k} \]

Read from spreadsheet data that $\phi P_{u} = 1,746$ k and use Eq. (1):
\[ \frac{1}{\phi \delta_{si}} = \frac{1}{\phi P_{x}} + \frac{1}{\phi P_{y}} - \frac{1}{\phi P_{wy}} \]
\[ = \frac{1}{764} + \frac{1}{877} - \frac{1}{1,746} \]
\[ = 0.00188, \text{ for which } \phi P_{wy} = 533 \text{ k} \]

Since 533 k is only 1.8% less than the required 543 k, the section is acceptable.

References

1. CRSI Design Handbook 2002, 9th Ed., Concrete Reinforcing Steel Institute, Schaumburg, IL.
2. ACI Design Handbook, SP-17(97), ACI Committee 340, American Concrete Institute, Farmington Hills, MI.
3. Furlong, Richard W., Basic Decisions for Designing Reinforced Concrete Structures, Prince, Davidson

Fig. 7 Interaction Diagram for Figure 3
Notes on Soft Metric Reinforcing Bars

It is important for readers of this document to be aware of current industry practice regarding soft metric reinforcing bars. The term "soft metric" is used in the context of bar sizes and bar size designations. "Soft metric conversion" means describing the nominal dimensions of inch-pound reinforcing bars in terms of metric units, but not physically changing the bar sizes. In 1997, producers of reinforcing bars (the steel mills) began to phase in the production of soft metric bars. The shift to exclusive production of soft metric bars has been essentially achieved. Virtually all reinforcing bars currently produced in the USA are soft metric. The steel mills’ initiative of soft metric conversion enables the industry to furnish the same reinforcing bars to inch-pound construction projects as well as to metric construction projects, and eliminates the need for the steel mills and fabricators to maintain a dual inventory.

The sizes of soft metric reinforcing bars are physically the same as the corresponding sizes of inch-pound bars. Soft metric bar sizes, which are designated #10, #13, #16, and so on, correspond to inch-pound bar sizes #3, #4, #5, and so on. The following table shows the one-to-one correspondence of the soft metric bar sizes to the inch-pound bar sizes. More information about soft metric reinforcing bars is given in Engineering Data Report No. 42, "Using Soft Metric Reinforcing Bars in Non-Metric Construction Projects". EDR No. 42 can be found on CRSI’s Website at www.crsi.org.

CRSI Website

Readers of this report are also encouraged to visit the CRSI Website for:

- Descriptions of CRSI publications and software, and ordering information
- Institute documents available for downloading
- Technical information on epoxy-coated reinforcing bars
- Technical information on continuously reinforced concrete pavement
- Membership in CRSI and member web links
- General information on the CRSI Foundation
- Information on the CRSI Design Awards Competition

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